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Multi-Dimensional *D*-Transform Approach to Modeling and Control of the Air Traffic Environment

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An elegant model of air traffic flow based on the extension of one-dimensional *D*-transforms is described. Multi-dimensional *D*-transforms of aircraft trajectories are derived and linearly combined to form the model of the air traffic environment. Unlike other aggregate models of air traffic flow, the *D*-transform model can preserve the trajectory details of individual aircraft. Moreover, it allows the algebraic treatment of aircraft delay and re-routing. The application of the *D*-transform models to air traffic flow control and other related air traffic management problems are discussed.

I. Introduction

Transform methods have proved to be powerful tools in automatic control and signal processing disciplines. The objective of this paper is to explore the application of *D*-transforms for modeling, analysis and control of air traffic. Methods from control theory are beginning to find applications in the air traffic management (ATM) systems. Recent research efforts have examined the problems of conflict resolution[1, 2], and air traffic flow control using Eulerian models[3, 4] and aggregate flow models[5 - 6]. These research efforts have the potential to produce revolutionary decision support tools for air traffic management, allowing for the systematic improvements in air traffic capacity and operational efficiency.

The motivation for the present research stems from the recent interest in methods for the derivation of air traffic flow models and control strategies. The traffic flow modeling approaches reported in References 3 - 6 have several weaknesses. Firstly, the Eulerian modeling technique[3, 4] and the aggregate modeling approach[5] do not preserve individual aircraft identities or the trajectory data. This fact makes it difficult to map flow control decisions derived using these models back to individual aircraft. Secondly, the arithmetic used in the modeling is based on real numbers, which can produce flow control decisions in terms of fractional aircraft counts. Finally, the averaging process used in the aggregate model[5, 6] results in infinite impulse response (IIR) models. Since the IIR models include both poles and zeros, they can artificially introduce issues regarding the "stability" into the air traffic flow models. What is required are models that preserve aircraft identities and routing information, base all the computations on integer arithmetic and do not introduce infinite impulse response approximations.

The present modeling approach views the airspace as a sequential finite state machine[7, 8], in which the aircraft are considered to be unique "bits" that are subject to sequential spatial and temporal shifts. Such a discretized point-of-view naturally leads to the description of the airspace using the *D*-transforms[7, 8].

Techniques such as Heaviside-Laplace transforms and *z*-transforms have greatly contributed to the systematic development of control theory and communications and signal processing theories. The transform approaches allow the characterization of the system dynamics by algebraic means. Consequently, design tasks can be formulated in terms of polynomial algebra. In addition to their analytical elegance, these transforms enable the application of powerful numerical linear-algebraic algorithms for the design and analysis. One-dimensional transforms have been popular in control theory and signal processing, while two-dimensional transforms have found applications in image processing.

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The *D*-transform can be considered to be a variation of the z-transform in which all the variables take-on integer values or numbers from the *Galois* field[7]. One-dimensional *D*-transform is extensively used in coding theory[9] and in the analysis of linear state machines. Over the past five decades, a wide variety of finite state machine design problems have been formulated in terms of the *D*-transform. Due to the emphasis on digital systems, a vast majority of the applications reported in the literature discuss the *D*-transforms in terms of the binary number system. This paper proposes the extension of the *D*-transform technique to multiple dimensions and models the air traffic environment using 4-dimensional *D*-transforms. The proposed approach allows the description of the air traffic environment in terms of multi-dimensional polynomials, which can be used for analysis and for deriving flow control strategies.

The starting point for the model is the predicted trajectories of aircraft. Software packages such as FACET[10] (Future ATM Concepts Evaluation Tool) or ACES[11] (Airspace Concept Evaluation System), which can provide trajectory predictions based on aircraft flight plans and the initial state of the air traffic environment are suitable for this purpose. The aircraft trajectories are discretized in time and space by a regular grid, or by any other spatio-temporal discretization. The *D*-transform of an aircraft trajectory in the airspace can then be written down by inspection. Due its linearity, the *D*-transform of individual aircraft trajectories can be linearly combined to form the overall *D*-transform model of the air traffic environment. By characterizing the dynamics of the air traffic flow in this manner, the *D*-transform approach creates an algebraic framework for analyzing the aircraft flow, and for designing flow control strategies. An interesting feature of the *D*-transform modeling approach is that it accommodates both Eulerian and Lagrangian modeling points-of-view[3, 4] in a single framework. Modeling the airspace using multi-dimensional D-transform will be discussed in the Section II.

The *D*-transform model of the air traffic environment can be used to address a variety of problems. Firstly, it provides a compact means for conveying the air traffic data. Secondly, it can be used for rapid assessment of the effects of aircraft time delay and re-routing. Thirdly, the transform model can be used to derive control strategies for modifying the traffic flow patterns in any desired manner. These strategies can be used as decision support tools or to automate air traffic flow control. The flow control using the *D*-transform model will be discussed in Section III. Conclusions and future work are given in Section IV.

II. Airspace Modeling using Four-Dimensional *D*-Transforms

Consider the discretization of the four dimensional space-time continuum as:

$$t = k\Delta t \qquad x = l\Delta x \qquad y = m\Delta y \qquad z = n\Delta z$$

The set of integers (k,l,m,n) discretize the space-time coordinates. Each aircraft can be conceptualized as a *Kronecker delta* sequence $\delta(k,l,m,n)$ in this four-dimensional space-time:

$$A(k,l,m,n) = \delta_{k_1,l_1,m_1,n_1}(k,l,m,n)\begin{cases} 1, & k = k_1, l = l_1, m = m_1, n = n_1 \\ 0, & k \neq k_1, l \neq l_1, m \neq m_1, n \neq n_1 \end{cases}$$

In order to preserve the identities of the aircraft, each aircraft can be denoted by the symbol A_i with $i=1, 2\ldots j$, with j being the total number of aircraft in the airspace. Wherever the model is used for the analysis of traffic flow wherein the individual aircraft identities are not important, these symbols can be replaced with unity. The airspace is made up of several such kronecker deltas, and their motion in the 4-dimensional space-time can be expressed through changes in their space-time indices (k,l,m,n). Following the image processing literature, elements of the discretized space will be termed as *Voxels* in this paper.

This conceptualization of individual aircraft can be made amenable to further analysis through the use of transform methods, as will be discussed in the following. As in the case of Heaviside-Laplace transforms and *z*-tranforms which enable the treatment of continuous and discrete time dynamic systems using algebraic methods, the *D*-transform allows the algebraic treatment of the aircraft trajectories and the traffic flow in the airspace. Note that the one-dimensional *D*-transform is routinely employed in communication theory[9].

Real and complex numbers that are employed in the analysis of linear continuous systems are examples of infinite fields. Since the number of aircraft in the airspace can take on only a finite number of values, they must be characterized with *finite fields*. These finite fields are called *Galois fields*[7] after the French mathematician who first investigated their properties. The *D*-transform can be conceptualized as the *z*-transform specialized to Galois functions.

The *D*-transform is defined for functions that are zero for negative values of the independent variable. For instance, for functions that depend on time, the *D*-transform is defined only for $0 \le t$. Let $g(k)$ be a sequence that takes on values from the *Galois* field. One-dimensional *D*-transform of this function is defined as[7]:

$$D[g(k)] = g(0) + g(1)D + g(2)D^2 + g(3)D^3 \ldots = \sum_{k=0}^{\infty} g(k)D^k$$

The symbol *D* can be thought of as a "delay" operator if the independent variable is discretized time, or a "shift operator" if the independent variable is a discretized spatial coordinate. It can be verified that the *D*-transform satisfies all the conditions for a linear transform.

Next, the one-dimensional *D*-transform can be extended to multiple dimensions using the analogy with multi-dimensional *z*-transforms employed in image processing[12]. For instance, the four-dimensional *D*-transform useful in the description of aircraft trajectories in the discretized space-time can be defined as:

$$D[g(k,l,m,n)] = \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} g(k,l,m,n) D_t^k D_x^l D_y^m D_z^n$$

The subscripts denote the independent variables, the time *t* and the three spatial coordinates *x, y, z*. Since the space under consideration is bounded and the aircraft trajectories end over finite intervals of time, for all practical purposes, the infinite sum can be replaced by a finite sum. Note that the multi-dimensional *D*-transform proposed here has not been previously discussed in the literature.

Define the origin of the space-time as 0^{th} hour Universal Time and the South-West corner of the *US* airspace on the latitude-longitude plane at the mean sea level. The instantaneous location of an aircraft *i* with respect the discretized space-time can be described using the *D*-transform as:

Aircraft Location: $A_i D_t^k D_x^l D_y^m D_z^n$

The integers *k, l, m, n* are the space-time indices locating the aircraft. The 4-dimensional *D*-transform can be used for describing the trajectory of an aircraft in the airspace. For instance, if the aircraft moved to a new position $2\Delta x, 3\Delta y, 4\Delta z$ from its current position during the next sample time, its *D*-transform is given by:

$$A_i D_t^{k+1} D_x^{l+2} D_y^{m+3} D_z^{n+4}$$

Equivalently,

$$A_i \left(D_t^k D_x^l D_y^m D_z^n\right)\left(D_t^1 D_x^2 D_y^3 D_z^4\right)$$

The *D*-transform of the aircraft trajectory over these two samples is then given by:

$$A_i D_t^k D_x^l D_y^m D_z^n \left(1 + D_t^1 D_x^2 D_y^3 D_z^4\right)$$

A simple airspace will be set up in the following to demonstrate the use of *D*-transforms for characterizing the air traffic flow. Consider an airspace discretized by nine cells at two altitudes, as illustrated in Figure 1. The numbering convention for each volumetric cell or *Voxel* is also indicated in Figure 1.

The trajectory of an aircraft A_1 that took-off from an airport in Voxel *(1,1,1)* at time sample 5 and landed at an airport in the Voxel *(1,3,1)* is indicated in Figure 1. For the sake of the present discussions, it is assumed that the aircraft transitions from one Voxel to the next in a unit time step. The *D*-transform of the aircraft trajectory is then given by:

$$A_1 \Big[D_t^5 D_x^1 D_y^1 D_z^1 + D_t^6 D_x^1 D_y^1 D_z^2 + D_t^7 D_x^2 D_y^1 D_z^2 + D_t^8 D_x^2 D_y^2 D_z^2 + D_t^9 D_x^1 D_y^2 D_z^2 + D_t^{10} D_x^1 D_y^3 D_z^2 + D_t^{11} D_x^1 D_y^3 D_z^1\Big]$$

or

$$A_1 D_t^5 D_x^1 D_y^1 D_z^1 \left[1 + D_t D_z + D_t^2 D_x D_z + D_t^3 D_x D_y D_z + D_t^4 D_y D_z + D_t^5 D_y^2 D_z + D_t^6 D_y^2 D_z\right]$$

In the *D*-transform approach, the trajectory can be conceptualized as a transfer function that operates on the initial location of the aircraft. If the aircraft is delayed at the departing airport by two time samples, the *D*-transform of the delayed trajectory can be obtained by multiplying the above *D*-transform by D_t^2 , producing:

$$A_1 D_t^7 D_x^1 D_y^1 D_z^1 \left[1 + D_t D_z + D_t^2 D_x D_z + D_t^3 D_x D_y D_z + D_t^4 D_y D_z + D_t^5 D_y^2 D_z + D_t^6 D_y^2 D_z\right]$$

Introduction of an en-route delay can also be modeled in a direct manner. For instance, in addition to the departure delay, if the aircraft is delayed by one time step when it is in Voxel *(2,2,2)*, its *D*-transform will be of the form:

$$A_1 D_t^7 D_x^1 D_y^1 D_z^1 \left[1 + D_t D_z + D_t^2 D_x D_z + D_t \left(D_t^3 D_x D_y D_z + D_t^4 D_y D_z + D_t^5 D_y^2 D_z + D_t^6 D_y^2 D_z\right)\right]$$

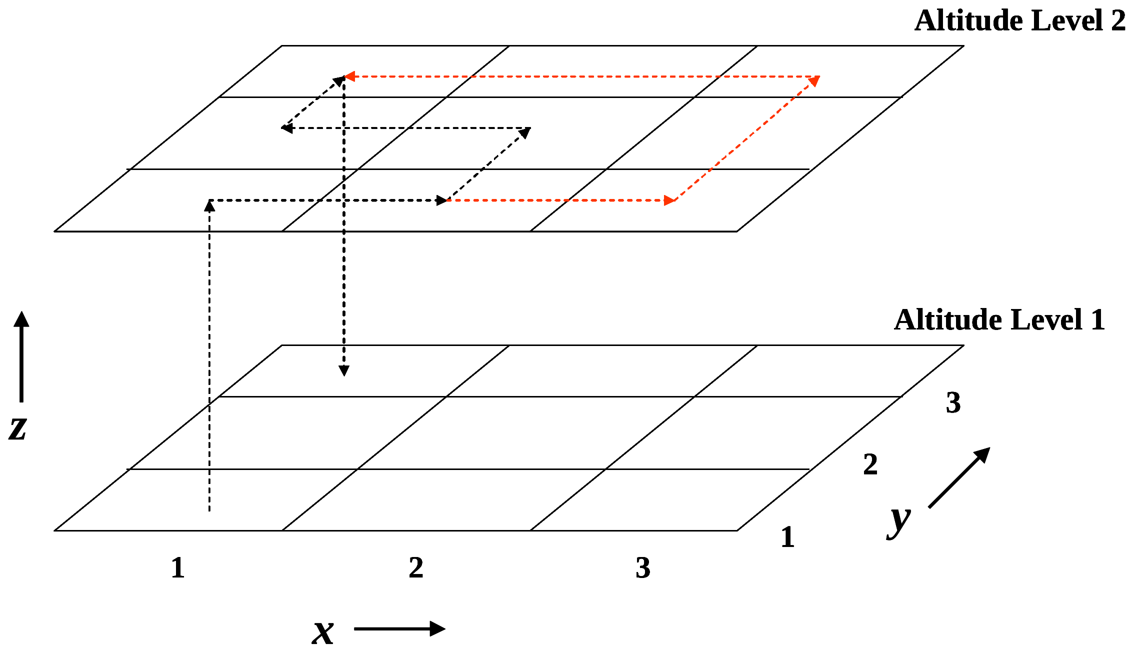

Figure 1. Trajectory of an Example Aircraft

In a manner similar to the modeling of the departure or en-route delay, the *D*-transform approach can be used to model trajectory rerouting. However, this is a more complex operation since it will involve simultaneous shift operations in both spatial and temporal independent variables. For instance, consider the rerouting of the aircraft along the trajectory marked with red in Figure 1. The *D*-transform of the rerouted aircraft trajectory can be modeled as:

$$\begin{aligned} A_1 D_t^5 D_x^1 D_y^1 D_z^1 \Big[&1 + D_t D_z + D_t^2 D_x D_z + \{D_x D_y^{-1}\} D_t^3 D_x D_y D_z + \{D_x^2\} D_t^4 D_y D_z \\ &+ \{D_x^2\} D_t^5 D_y^2 D_z + \{D_x D_z\} D_t^6 D_y^2 D_z + \{D_t^7 D_y^2 D_z + D_t^8 D_y^2\}\Big] \end{aligned}$$

The quantities within the braces perform the rerouting function. Note that the above operations assumed that a spatial grid can be traversed in a single time unit.

The foregoing development illustrated the *D*-transform modeling of an aircraft trajectory. If multiple aircraft are present in the environment, their *D*-transforms can be linearly combined to form the *D*-transform model of the air traffic environment. This representation can then be used for aircraft flow analysis and for the design of control strategies to meet desired flow objectives.

In order to illustrate the derivation of the *D*-transform of the airspace, a highly simplified airspace shown in Figure 2 is next considered. The traffic consists of four aircraft, with aircraft A_1 and A_2, taking off from an airport in Voxel *(1,1,1)* two time samples apart, and landing at airports *(1,3,1)* and *(3,3,1)* respectively. The aircraft A_3 takes-off from an airport in *(3,3,1)* and lands in the airport in *(1,3,1)* one time step later than the aircraft A_1. A_4 is an en-route aircraft that enters the airspace two time steps after the takeoff of the aircraft A_1 and stays in the airspace for three time steps. The *D*-transform of the individual aircraft trajectories can be written-down by inspection as:

$$\text{A}_1\text{:}\ A_1\left[D_t^1 D_x^1 D_y^1 D_z^1 + D_t^2 D_x^1 D_y^1 D_z^2 + D_t^3 D_x^2 D_y^1 D_z^2 + D_t^4 D_x^2 D_y^2 D_z^2 + D_t^5 D_x^1 D_y^2 D_z^2 + D_t^6 D_x^1 D_y^3 D_z^2 + D_t^7 D_x^1 D_y^3 D_z^1\right]$$

$$\text{A}_2\text{:}\ A_2\left[D_t^3 D_x^1 D_y^1 D_z^1 + D_t^4 D_x^1 D_y^1 D_z^2 + D_t^5 D_x^2 D_y^1 D_z^2 + D_t^6 D_x^3 D_y^1 D_z^2 + D_t^7 D_x^3 D_y^2 D_z^2 + + D_t^8 D_x^3 D_y^3 D_z^2 + D_t^9 D_x^3 D_y^3 D_z^1\right]$$

$$\text{A}_3\text{:}\ A_3\left[D_t^2 D_x^3 D_y^3 D_z^1 + D_t^3 D_x^3 D_y^3 D_z^2 + D_t^4 D_x^2 D_y^3 D_z^2 + D_t^5 D_x^2 D_y^2 D_z^2 + D_t^6 D_x^1 D_y^2 D_z^2 + D_t^7 D_x^1 D_y^3 D_z^2 + D_t^8 D_x^1 D_y^3 D_z^1\right]$$

$$\text{A}_4\text{:}\ A_4\left[D_t^3 D_x^1 D_y^2 D_z^2 + D_t^4 D_x^2 D_y^2 D_z^2 + D_t^5 D_x^3 D_y^2 D_z^2\right]$$

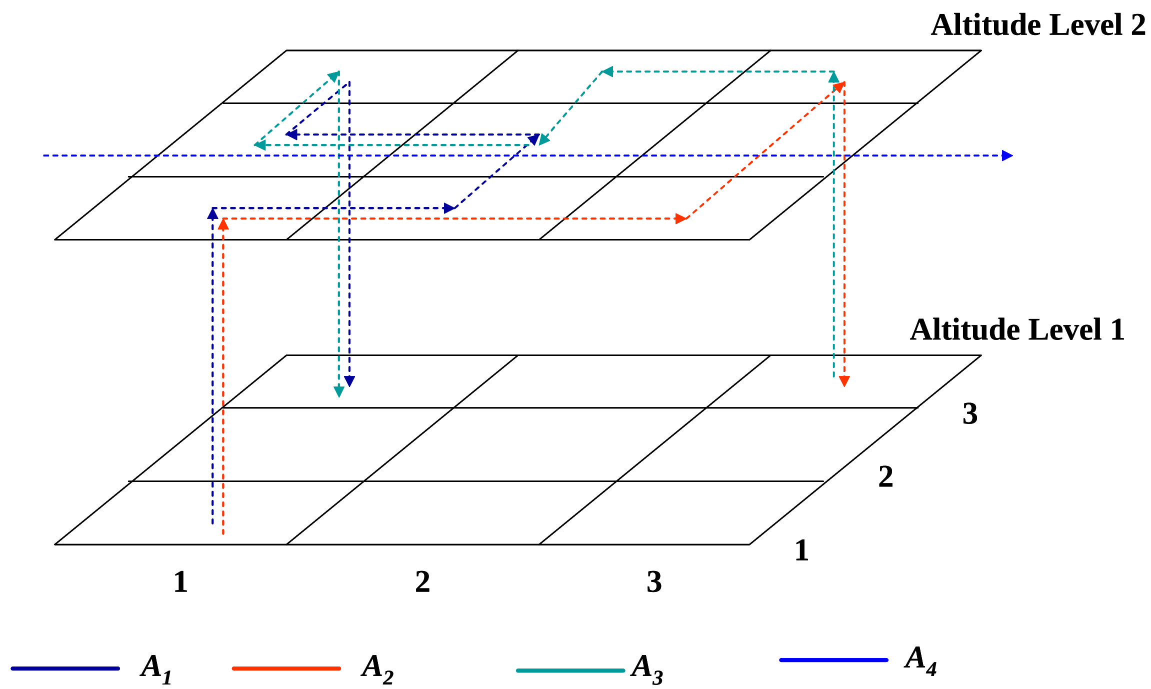

Figure 2. Trajectories of Aircraft in the Example Airspace

Since the *D*-transforms of aircraft trajectories are given by polynomials in terms of the transform variable *D*, they can be combined to obtain the *D*-transform of the airspace as:

$$\begin{aligned} & D_x^1 D_y^1 D_z^1\left(A_1 D_t + A_2 D_t^3\right) + D_x^3 D_y^2 D_z^1\left(A_3 D_t^2 + A_2 D_t^6\right) + D_x^1 D_y^3 D_z^1\left(A_1 D_t^7 + A_3 D_t^8\right) \\ & + D_x^1 D_y^1 D_z^2\left(A_1 D_t^2 + A_2 D_t^4\right) + D_x^2 D_y^1 D_z^2\left(A_1 D_t^3 + A_2 D_t^5\right) + D_x^3 D_y^1 D_z^2\left(A_2 D_t^6\right) \\ & + D_x^1 D_y^2 D_z^2\left(A_4 D_t^3 + A_1 D_t^5 + A_3 D_t^6\right) + D_x^2 D_y^2 D_z^2\left(\{A_1 + A_4\} D_t^4 + A_3 D_t^5\right) \\ & + D_x^3 D_y^2 D_z^2\left(A_4 D_t^5 + A_2 D_t^7\right) + D_x^1 D_y^3 D_z^2\left(A_1 D_t^6 + A_3 D_t^7\right) + D_x^2 D_y^3 D_z^2\left(A_3 D_t^4\right) \\ & + D_x^3 D_y^3 D_z^2\left(A_3 D_t^3 + A_2 D_t^8\right) \end{aligned}$$

The above expression collects together the aircraft moving through each of the voxels in the example airspace, and can be used to quickly visualize the nature of traffic flow at each voxel in the airspace. For instance, examination of the quantities within the parenthesis reveals that the highest traffic density occurs in Voxel *(2,2,2)* at the 4th temporal sample.

The *D*-transform of the airspace can be rearranged in a several different ways. For instance, the terms in the airspace *D*-transform can be rearranged as coefficients of the spatial independent variables *x, y* or *z*, to get an idea about the spatial distribution of traffic. If individual aircraft identities are not of interest, the symbols A_1 through A_4 can be replaced with unity, resulting in the *D*-transform of the airspace as:

$$D_x^1 D_y^1 D_z^1\left(D_t + D_t^3\right) + D_x^3 D_y^2 D_z^1\left(D_t^2 + D_t^6\right) + D_x^1 D_y^3 D_z^1\left(D_t^7 + D_t^8\right) + D_x^1 D_y^1 D_z^2\left(D_t^2 + D_t^4\right) + D_x^2 D_y^1 D_z^2\left(D_t^3 + D_t^5\right)$$
$$+ D_x^3 D_y^1 D_z^2\left(D_t^6\right) + D_x^1 D_y^2 D_z^2\left(D_t^3 + D_t^5 + D_t^6\right) + D_x^2 D_y^2 D_z^2\left(2\,D_t^4 + D_t^5\right) + D_x^3 D_y^2 D_z^2\left(D_t^5 + D_t^7\right) + D_x^1 D_y^3 D_z^2\left(D_t^6 + D_t^7\right)$$
$$+ D_x^2 D_y^3 D_z^2\left(D_t^4\right) + D_x^3 D_y^3 D_z^2\left(D_t^3 + D_t^8\right)$$

Relationships between traffic flows at different points in the airspace are immediately apparent from the above *D*-transform. For instance, if the relationship between the traffic flows at the airports in voxels *(1,1,1), (1,3,1)* and the en-route voxel *(2,2,2)* are desired, the corresponding coefficients can be gathered and arranged as follows:

Traffic Flow at airport in the Voxel *(1,1,1)*: $F_{1,1,1} = \left(A_1 D_t + A_2 D_t^3\right)$

Traffic flow at Voxel *(2,2,2)*: $F_{2,2,2} = \left(\{A_1 + A_4\} D_t^4 + A_3 D_t^5\right)$

Traffic Flow at the airport in the Voxel *(1,3,1)*: $F_{1,3,1} = \left(A_1 D_t^7 + A_3 D_t^8\right)$

From these expressions, relationship between the traffic flows at Voxel *(1,1,1)* and Voxel *(2,2,2)* can be expressed as:

$$F_{2,2,2} = D_t^3 F_{1,1,1} - A_2 D_t^6 + A_4 D_t^4 + A_3 D_t^5$$

In order to be consistent with the properties of finite fields[7], it is important to note that the coefficient $-A_2$ should be interpreted as the *additive inverse* of A_2. Similarly, the relationship between the traffic flows at Voxel *(1,1,1)* and Voxel *(1,3,1)* is given by:

$$F_{1,3,1} = F_{1,1,1} D_t^6 - A_2 D_t^9 + A_3 D_t^8$$

As before, the coefficient $-A_2$ should be interpreted as the *additive inverse* of A_2. Both these flow relationships illustrate the fact that aircraft arriving at a particular destination may have originated at multiple points. Consequently, complete control of the air traffic flow may not be achievable without applying control actions at all these originating points. Note that the traffic flow relationships are dependent on the time interval under consideration.

The expressions for traffic flows at different voxels in the airspace can also be rearranged in a vector-matrix form to get a better idea of the traffic flow:

$$\begin{bmatrix} F_{1,1,1} \\ F_{2,2,2} \\ F_{1,3,1} \end{bmatrix} = \begin{bmatrix} D_t & D_t^3 & 0 & 0 \\ D_t^4 & 0 & D_t^5 & D_t^4 \\ D_t^7 & 0 & D_t^8 & 0 \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \\ A_3 \\ A_4 \end{bmatrix}$$

When arranged in this form, the relative influence of individual aircraft on air traffic flows at various voxels can be immediately visualized. The effect of delaying one or more aircraft on the traffic flow at one or more voxels can also be readily evaluated using this matrix representation.

2.1. Center-Level *D*-Transform Model

The *D*-transform model of the airspace at Air Traffic Control Center-level can be derived by assuming the spatial discretization in terms of Centers instead of employing a regular grid in the *x*, *y*, *z* spatial coordinates. As in the previous case, the aircraft trajectories are first expressed in terms of the Air Traffic Control Centers, and the *D*-

transforms derived. These are then be linearly combined to obtain the Center-level *D*-transform model of the airspace. Interestingly, this modeling approach process can be applied to any spatial discretization of the airspace.

This section will illustrate the *D*-transform modeling approach for the continental US Center-level airspace discretization given in Figure 3, reproduced from Reference 5. Figure 3 illustrates trajectories of four aircraft trajectories given in terms of the Centers they fly through. Their flight plans are:

A_1: ZOA-ZLC-ZMP-ZAU-ZOB
A_2: ZDV-ZMP-ZAU-ZID
A_3: ZFW-ZME-ZID-ZDC-ZNY
A_4: ZMA-ZJX-ZDC-ZOB

For simplicity, it is assumed that all four aircraft takeoff at the same time, and that the aircraft transitions from one Center to the next in a time unit. Note that if the exact transition times between Centers are known, perhaps from an airspace simulation, these can be readily incorporated in the model. Moreover, in the interests of maintaining simplicity, it is assumed that every aircraft takes-off from an airport in a Center and reaches the cruise flight altitude in one time unit. With this, the *D*-transforms of the aircraft trajectories can be written down by inspection as:

$$A_1\left[D_t D_{ZOA} D_z + D_t^2 D_{ZOA} D_z^2 + D_t^3 D_{ZLC} D_z^2 + D_t^4 D_{ZMP} D_z^2 + D_t^5 D_{ZAU} D_z^2 + D_t^6 D_{ZOB} D_z^2 + D_t^7 D_{ZOB} D_z\right]$$
$$A_2\left[D_t D_{ZDV} D_z + D_t^2 D_{ZDV} D_z^2 + D_t^3 D_{ZMP} D_z^2 + D_t^4 D_{ZAU} D_z^2 + D_t^5 D_{ZID} D_z^2 + D_t^6 D_{ZDC} D_z^2 + D_t^7 D_{ZDC} D_z\right]$$
$$A_3\left[D_t D_{ZFW} D_z + D_t^2 D_{ZFW} D_z^2 + D_t^3 D_{ZME} D_z^2 + D_t^4 D_{ZID} D_z^2 + D_t^5 D_{ZDC} D_z^2 + D_t^6 D_{ZNY} D_z^2 + D_t^7 D_{ZNY} D_z\right]$$
$$A_4\left[D_t^1 D_{ZMA} D_z + D_t^2 D_{ZMA} D_z^2 + D_t^3 D_{ZJX} D_z^2 + D_t^4 D_{ZDC} D_z^2 + D_t^5 D_{ZOB} D_z^2 + D_t^6 D_{ZOB} D_z\right]$$

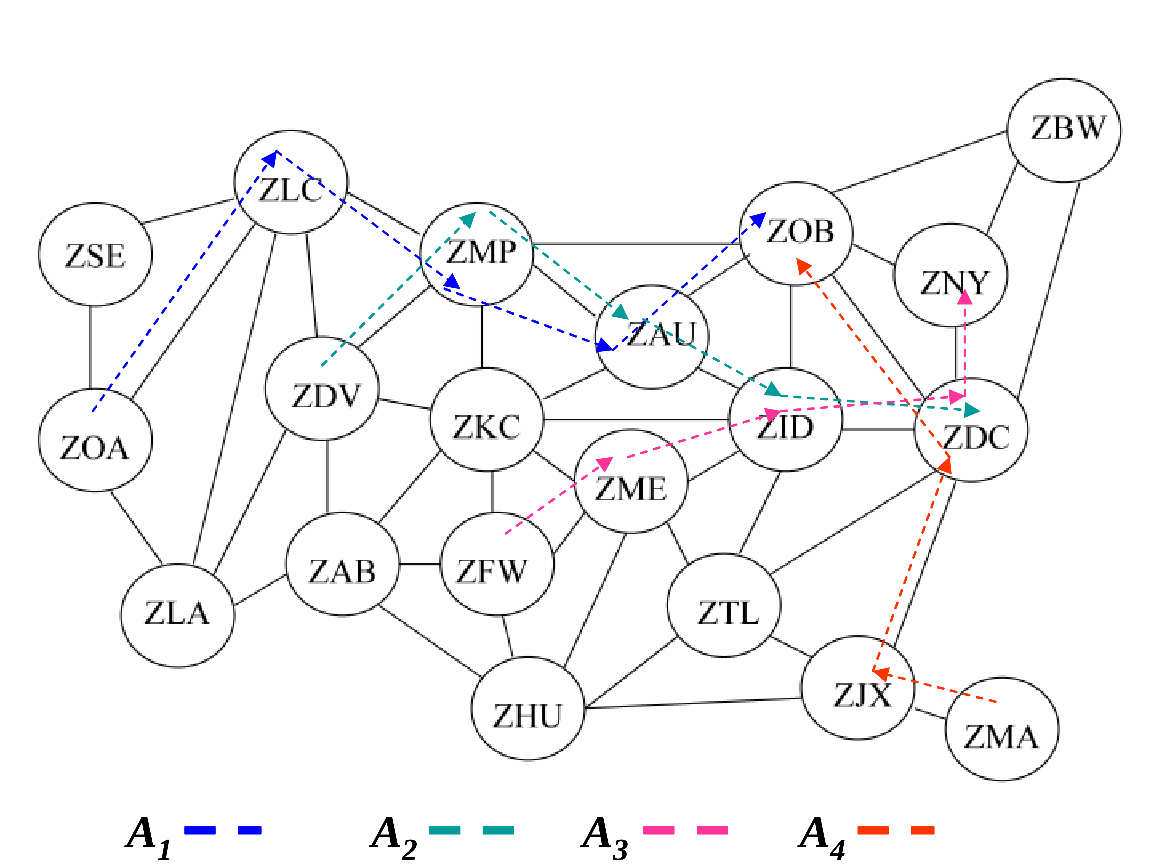

Figure 3. Aircraft Trajectories in the Center-Level Airspace

The *D*-transform of aircraft trajectories can be next be linearly combined to derive the Center-level *D*-transform model of the airspace as:

$$D_t D_z\left(A_1 D_{ZOA} + A_2 D_{ZDV} + A_3 D_{ZFW} + A_4 D_{ZMA}\right) + D_t^2 D_z^2\left(A_1 D_{ZOA} + A_2 D_{ZDV} + A_3 D_{ZFW} + A_4 D_{ZMA}\right)$$
$$+ D_t^3 D_z^2\left(A_1 D_{ZLC} + A_2 D_{ZMP} + A_3 D_{ZME} + A_4 D_{ZJX}\right) + D_t^4 D_z^2\left(A_1 D_{ZMP} + A_2 D_{ZAU} + A_3 D_{ZID} + A_4 D_{ZDC}\right)$$
$$+ D_t^5 D_z^2\left(A_1 D_{ZAU} + A_2 D_{ZID} + A_3 D_{ZDC} + A_4 D_{ZOB}\right) + D_t^6 D_z^2\left(A_1 D_{ZOB} + A_2 D_{ZDC} + A_3 D_{ZNY}\right) + D_t^6 D_z A_4 D_{ZOB}$$
$$+ D_t^7 D_z\left(A_1 D_{ZOB} + A_2 D_{ZDC} + A_3 D_{ZNY}\right)$$

As in the case of the *D*-transform model of the airspace on uniform spatio-temporal discretization, the traffic at any spatial location can be obtained by collecting the coefficients of a particular Center. For instance,

Traffic Flow at the Center ZAU, at the altitude level 2: $D_t^4 A_2 + D_t^5 A_1$

Traffic Flow at the Center ZDC, at the altitude level 2: $D_t^4 A_4 + D_t^5 A_3 + D_t^6 A_2$

Traffic Flow at the airport in the Center ZOB: $D_t^6 A_4 + D_t^7 A_1$

If aircraft identities are not important, the symbols A_1, A_2, A_3, A_4 can be replaced with unity to obtain the time histories of aircraft flow through the specified spatial locations.

As stated at the beginning of this section, *D*-transform models can be derived for any defined spatial discretization schemes. For instance, the methodology can be adapted for generating Sector-level and Terminal Area-Level *D*-Transform models.

A. Inverse *D*-Transform

Just as the direct transform, the inverse *D*-transform can be derived by inspection. Basically, the exponents of the *D*-operators must be interpreted as the sample indices. For instance, the inverse *D*-transform of the traffic flow at the Center ZDC: $D_t^4 A_4 + D_t^5 A_3 + D_t^5 A_2$ is given by:

$$0(0), 0(\Delta t), 0(2\Delta t), 0(3\Delta t), A_4(4\Delta t), A_3(5\Delta t), A_2(6\Delta t), 0(7\Delta t), 0(8\Delta t)...$$

Note that the result is in the form of a one-dimensional array, with the array index denoting the sample number. In a similar manner, the inverse transforms of two-dimensional *D*-transform models can be written in the form of two-dimensional arrays, with the array indices denoting the sample numbers of the independent variables. However, beyond two dimensions, it is awkward to express the inverse transforms due the need for multi-dimensional arrays.

It may be observed from the foregoing that the *D*-transform modeling methodology allows the description of traffic flows in the airspace in terms of algebraic expressions. These expressions can be manipulated in a variety of ways to derive analytical insights. Moreover, they can be readily employed for designing flow control strategies, as will be indicated in the following section.

Due to the systematic manner in which the *D*-transform models can be derived from discretized aircraft trajectories, the *D*-transform methodology is amenable to computer implementation. At the time of this writing, a software package has been developed that automatically derives *D*-transform models of airspace from trajectory predictions generated using the FACET[10] software. This package can also provide inverse *D*-transforms. This software is currently being adapted for enabling interactive design of air traffic flow control strategies.

III. Air Traffic Flow Control using D-Transform Models

As indicated in the foregoing section, the *D*-transform models are useful for analyzing the dynamics of the traffic flow in the airspace, and for assessing the effects of delaying and rerouting aircraft on air traffic flow. This section will illustrate how the *D*-transform models can be used to design flow control strategies. Due to the preliminary nature of the present research, discussions will be provided with respect to a simple flow control situation.

The types of flow control employed in the air traffic management systems are:

1. Ground delay and ground stops at the departure airport(s) to control en-route congestion or traffic flow rate at the arrival airports.
2. Traffic flow metering en-route to control congestion at designated regions of the airspace or arrival rates at the airports.
3. Rerouting air traffic around regions of severe weather or restricted airspace while ensuring that the traffic densities remain at desired levels.

In the first two cases, the primary control mechanism is the delay of aircraft, while the control mechanism in 3 involves shaping the trajectory. The discussions in the present paper will primarily focus on delaying the aircraft to achieve control objectives. The flow control problem involving rerouting will be addressed in future research efforts.

Consider the relationship between the traffic flow in Voxel *(1,1,1)* and *(2,2,2)* in the airspace model based on uniform spatial discretization described in Section II. For the sake of clarity, these are reproduced in the following:

Traffic Flow at airport in the Voxel *(1,1,1)*: $F_{1,1,1} = \left(A_1 D_t + A_2 D_t^3\right)$

Traffic flow at Voxel *(2,2,2)*: $F_{2,2,2} = \left(\{A_1 + A_4\} D_t^4 + A_3 D_t^5\right)$

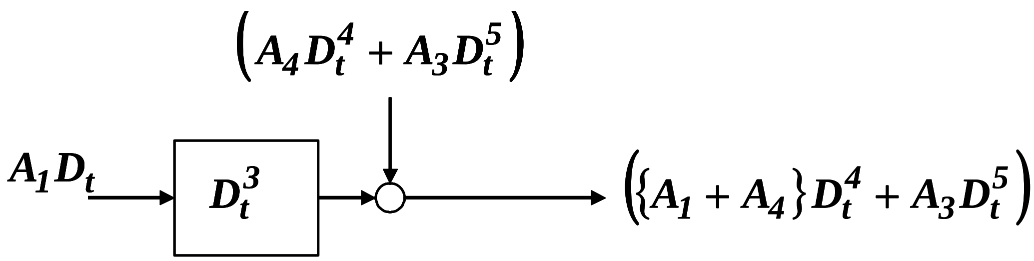

Figure 4. Relationship between Traffic Flows at Voxel *(1,1,1)* and Voxel *(2,2,2)*

It can be immediately observed that the traffic flow at Voxel *(2,2,2)* cannot be completely controlled by delaying aircraft at Voxel *(1,1,1)*. If the aircraft departure at Voxel *(1,1,1)* can be delayed by various amounts, the achievable traffic flow rates at Voxel *(2,2,2)* can be found to be:

Control at Voxel *(1,1,1)*	**Traffic Flow History at Voxel *(2,2,2)***
No Delay: D_t^0	$2\,D_t^4 + D_t^5$
Unit Delay: D_t^1	$D_t^4 + 2D_t^5$
Delay by Two Units: D_t^1	$D_t^4 + D_t^5 + D_t^6$
Delay by Three Units: D_t^1 …	$D_t^4 + D_t^5 + D_t^7$ …

The above table clearly illustrates all the achievable flow rate histories at Voxel *(2,2,2)* by controlling the departure delay at Voxel *(1,1,1)*. It may be observed that the aircraft departing Voxel *(1,1,1)* must be delayed by two time units to achieve uniform flow rate at Voxel *(2,2,2)*. However, the same uniform flow rate can be achieved if the aircraft A_3 were delayed by one time unit at its departure airport in Voxel *(3,3,1)*, and the aircraft at Voxel *(1,1,1)* is delayed by one time unit. Finally, the same flow objective can be achieved by delaying the en route aircraft A_4 by one time unit, together with a single time unit delay at Voxel *(1,1,1)* or *(3,3,1)*.

It may be observed that the algebraic nature of the *D*-transform provides directions for flow control. A similar analysis can be undertaken for flow control problems involving aircraft rerouting. However, exhaustive analysis such as the one discussed in the foregoing may not be feasible under more complex flow histories. In those cases, formal polynomial factorization methods may be useful in deriving flow control strategies. Future research will address these and other related flow control problems.

IV. Conclusions

This paper discussed a new approach to modeling and control of air traffic flow using the multi-dimensional *D*-transforms. It is based on extending the one-dimensional D-transform theory to four dimensions. The *D*-transform approach provides an algebraic framework for describing the air traffic environment, and yields insights about the nature of traffic flows at various points-of-interest in the airspace.

The *D*-transforms of individual aircraft trajectories are determined from trajectory predictions. These are then linearly combined to derive the *D*-transform of the airspace. The effect of delaying aircraft and rerouting on air traffic flows at specific locations in the airspace can be determined through algebraic manipulation of the airspace

D-transform. The use of the airspace *D*-transform for air traffic flow control was discussed. It was shown that the methodology allows the examination of various control possibilities using polynomial manipulations.

In addition to being useful for describing and manipulating air traffic flow, the *D*-transform technique has potential applications in other trajectory modeling and manipulation problems such as mission design for multiple cooperating vehicles.

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