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Multi-Stepping Solution to Linear Two Point Boundary Value Problems in Missile Integrated Control

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A multi-stepping state transition matrix approach for solving a linear two-point boundary value problem is developed. The algorithm employs partitioned state transition matrix of the Hamiltonian system, and is computationally less expensive than backward integration of differential Riccati equation. This fact makes it ideally suited for online implementation. The application of this technique is illustrated for a finite interval moving mass actuated missile guidance-autopilot for target interception. A combination of feedback linearization and the multi-stepping linear boundary value solution algorithm is employed in the example. Closed loop simulation results are given.

I. Introduction

Two-point boundary value problems result from the application of optimal control theory to missile guidance problems. A special two-point boundary value problem of interest results from a linear dynamic system while optimizing a quadratic performance index consisting of states and controls[1 - 3]. As an example, these problems arise in the derivation of guidance-control laws that minimize the terminal miss distance, while penalizing the control effort. Different techniques for solving the two-point boundary value problem are discussed in Reference 1. Techniques such as the shooting method require the solution to the initial value problem using either numerical forward integration of the differential equations or the use of state-transition matrix solution. The numerical integration of differential equations is a time consuming method, while the state-transition matrix approach suffers from numerical difficulties[1] for large time intervals and ill-conditioned Hamiltonian matrices. Control computation for a finite-interval LQR problem can also be posed as a solution to the Riccati differential equation. However, integrating the differential Riccati equation at each instant of time backwards may not feasible for real-time control computation. Off-line gain computation and implementation requires large memory from the on-board processor. Moreover, the solution may not be useful in a dynamic setting where the boundary conditions keep changing with time.

An analytical state transition matrix based solution has been discussed in Reference 1. However, the state-transition matrix can be difficult to compute for large time periods. This paper addresses the numerical ill conditioning problem by dividing the time-interval into multiple intervals and employing the transition matrix solution in each subinterval. This approach dramatically improves the numerical condition of the problem, while avoiding the need for numerical integration. One of the byproducts of this algorithm is the solution to the differential Riccati equation. The technique can be implemented using linear-algebraic operations available in software packages such as *LAPACK*[4]. The numerical algorithm is described in the Section II. Section III describes the application of this technique to the development of a finite-interval guidance-control system for a kinetic warhead. Engagement simulation results for a moving mass actuated kinetic warhead are presented in section IV. Section V summarizes the conclusions from the present research.

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II. Multi-Stepping Technique

The two-point boundary value problem resulting from control computation of a linear dynamic system while optimizing a quadratic performance index can be defined as follows:

$$\dot{z} = Az + Bu$$

$$\min_{u} J = z_f^T S_f z_f + \int_0^{t_f} (z^T Qz + u^T Ru)dt \tag{1}$$

$$\text{subject to } z(0) = z_0 \quad and \quad z(t_f) = z_f$$

The optimal control *u* for the above problem is a function of the co-state vector at the initial λ *(0)*, which can be obtained by solving the following two-point boundary value problem:

$$\begin{bmatrix} \dot{z} \\ \dot{\lambda} \end{bmatrix} = \begin{bmatrix} A & -BR^{-1}B^T \\ Q & -A^T \end{bmatrix} \begin{bmatrix} z \\ \lambda \end{bmatrix}, \quad z(0) = z_0, \quad \lambda(t_f) = S_f z(t_f) \tag{2}$$

The coefficient matrix in the above linear system of differential equations is known as the Hamiltonian matrix. The solution to the above two-point boundary value problem can be obtained if the state-transition matrix of the Hamiltonian matrix can be computed for time ($t = t_f$). However, in most problems, the state transition matrix can be difficult to compute for large values of *t* if the Hamiltonian matrix has eigen-values with positive real parts. The state transition matrix can also be difficult to compute for small values of *t* for an aggressive choice of *Q* and *R*, which can result in an ill-conditioned Hamiltonian matrix. However, for sufficiently small *t* the state transition matrix can be computed for any choice of *Q* and *R*. This fact forms the basis of the multiple stepping algorithm developed in this paper.

The interval [$0\ t_f$] is divided into sub-intervals of equal length 't_l' such as [$0\ t_l$], [$t_l\ 2t_l$],…….[$(n-1)t_l\ \ t_f$]. The length of the interval is chosen such that the state transition matrix be computable at $t = t_l$. The state transition matrix for each of these intervals is represented as shown below:

$$N = e^{Mt_l} = e^{M(t_2 - t_1)} = e^{M(t_1 + t_l - t_1)} = e^{Mt_l} = \ldots\ldots \tag{3}$$

where *M* is the Hamiltonian matrix, t_1 and t_2 are the time instants at the end of the first and second intervals respectively. The solution for state and co-state vector at end of each time instant can be represented as shown below:

$$\begin{bmatrix} z_1 \\ \lambda_1 \\ z_2 \\ \lambda_2 \\ . \\ . \\ z_f \\ \lambda_f \end{bmatrix} = \begin{bmatrix} N & 0 & 0 & 0 & 0 \\ 0 & N & 0 & 0 & 0 \\ . & . & . & . & . \\ . & . & . & . & . \\ 0 & 0 & 0 & 0 & N \end{bmatrix} \begin{bmatrix} z_0 \\ \lambda_0 \\ z_1 \\ \lambda_1 \\ . \\ . \\ z_{f-1} \\ \lambda_{f-1} \end{bmatrix} \tag{4}$$

The central idea of the multi-stepping approach is to solve for the initial condition on the co-state vector recursively using this matrix equation. The multi-stepping strategy for solving the initial condition on the co-state vector is demonstrated in a two interval setting below:

$$\begin{bmatrix} z_1 \\ \lambda_1 \\ z_2 \\ \lambda_2 \end{bmatrix} = \begin{bmatrix} N & 0 \\ 0 & N \end{bmatrix} \begin{bmatrix} z_0 \\ \lambda_0 \\ z_1 \\ \lambda_1 \end{bmatrix} \tag{5}$$

The last two equations (λ_2 and z_2) of the above system of equations can be expanded as shown below:

$$\begin{aligned} z_2 &= N_{11} z_1 + N_{12} \lambda_1 \\ \lambda_2 &= N_{21} z_1 + N_{22} \lambda_1 \end{aligned} \tag{6}$$

where N_{ij}'s are sub-matrices of the state transition matrix *N*. Invoking the boundary condition $\lambda_2 = S_f z_2$, the following equations are obtained:

$$\lambda_1 = [N_{22} - S_f N_{12}]^{-1} [S_f N_{11} - N_{21}] z_1 = S_1 z_1 \tag{7}$$

S_1 in the above expression can be evaluated since the state transition matrix *N* can be evaluated and therefore, its sub-matrices. Using the above result and solving the system of equations for λ_1, z_1 in the same fashion the following expression for λ_0 is obtained

$$\lambda_0 = [N_{22} - S_1 N_{12}]^{-1} [S_1 N_{11} - N_{21}] z_0 = S_0 z_0 \tag{8}$$

The above expression offers a computable solution to the initial condition on the co-state vector. The multi stepping strategy obtains the relation between the co-state vector λ and the state vector *z* at the end of each interval starting with S_f at $t = t_f$. It is well known that the matrix connecting these two vectors is the solution to the differential Riccati equation. Therefore, S_1 is the solution of the finite interval Riccati equation solution at time $t=t_1$. As the number of time intervals is increased the following recursive relation can be used to compute the Riccati equation solution(*S*) at different instances of time in between.

$$S_{i-1} = [N_{22} - S_i N_{12}]^{-1} [S_i N_{11} - N_{21}], \quad S(t = t_f) = S_f \tag{9}$$

and $\lambda_0 = S_0 z_0$

$$u(0) = -R^{-1} B^T \lambda(0). \tag{10}$$

The multi-stepping strategy is computationally much more efficient than the numerical backward integration of differential Riccati equation at each instant of time, since it uses the state transition matrix to propagate states and co-states. It also offers a solution when conventional techniques requiring solution to the initial value problem fail. *It should be noted that online implementation of the above technique only requires the storage of the matrix N on the onboard computer*.

In the following two sections the above technique will be applied to a finite-duration missile guidance problem.

III. Missile Guidance

Conventional missile control is typically accomplished using actuators like fins or reaction jets. Moreover the guidance and control problems are decoupled and addressed separately. The guidance algorithm generates the acceleration commands for the autopilot. The autopilot then tracks the guidance commands to achieve target interception. Recent research[5] has advanced a technique for deriving integrated guidance-autopilot system in a finite interval setting. This formulation of the integrated guidance-autopilot problem forms the basis for the present example.

A novel moving mass based actuation system was proposed in Reference 6. These actuators are completely enclosed within the envelope of the kinetic warhead (KW). An integrated approach to the guidance and control of the KW was also demonstrated in that paper. A combination of continuous time feedback linearization and pole placement technique was used for the integrated guidance-control system design (IGCSD). The controller's task was to regulate the instantaneous line of sight rate of the target with respect to the missile. A commercially available nonlinear control system design software[7,8] was been used for controller design.

An IGCSD formulation that is suitable for finite duration implementation will be developed in this section. Figure 1 shows a schematic of the kinetic warhead's target interception scenario. Interception occurs when the relative position vector between the KW and the target goes to zero. Therefore, the control objective is to drive the terminal relative position vector between the KW and target to zero. However, only two components of the relative vector are controllable. The component of the relative position vector along the longitudinal body axis is not controllable using the moving mass actuators. This leaves the mass movement along the body y-axis and the mass movement along the body z-axis as the control variables in the problem.

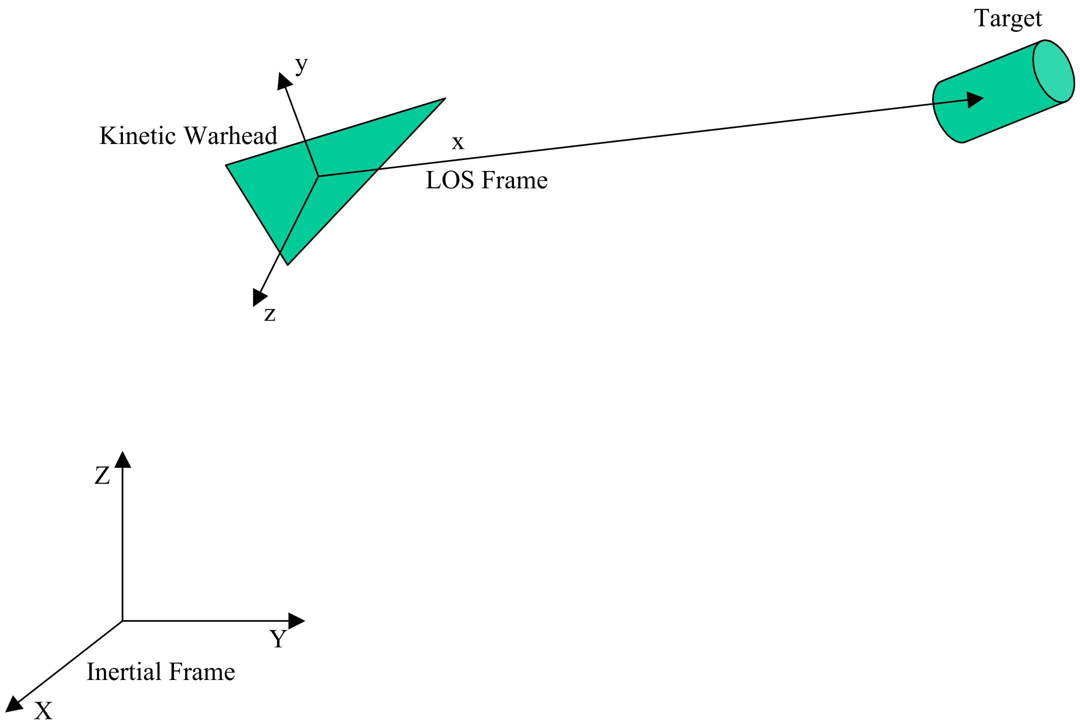

Figure 1 : Kinetic Warhead and Target Engagement Scenario

The control is achieved by generating acceleration components normal to the velocity vector. The interception problem is posed as a two parameter control problem in a plane normal to the line of sight vector. In this example the relative vector components along the y and z directions of a LOS frame will be controlled over a finite duration. The LOS frame is chosen such the x-axis coincides with the line of sight vector at the initial time. The components of the initial relative position vector Δ_y and Δ_z are zero in this frame of reference. The control chains[4] for the y and z actuators are shown below:

$$\begin{aligned} &\delta_{yc} \to u_y \to \dot{\delta}_y \to \delta_y \to r \to v \to \dot{\Delta}_y \to \Delta_y \\ &\delta_{zc} \to u_z \to \dot{\delta}_z \to \delta_z \to q \to w \to \dot{\Delta}_z \to \Delta_z \end{aligned} \tag{11}$$

δ_{yc} - Position Command to the y actuator

δ_{zc} - Position Command to the z actuator

$u_y = k_p(\delta_{yc} - \delta_y) - k_v\dot{\delta}_y + m_y a_y$ - Force applied on the y actuator,
k_p and k_v are the position servo gains
$u_z = k_p(\delta_{zc} - \delta_z) - k_v\dot{\delta}_z + m_z a_z$ - Force applied on the z actuator

δ_y - Actual Position of the y actuator

δ_z - Actual Position of the z actuator

a_y – Acceleration component along the body-frame y-axis
a_z – Acceleration component along the body-frame z-axis

m_y – mass of the y-actuator
m_z – mass of the z-actuator

q – Pitch rate of the KW
r – Yaw rate of the KW

v – Body-frame velocity component along the y-axis
w – Body-frame velocity component along the z-axis

The variables $\dot{\Delta}_y, \dot{\Delta}_z$ are the components of the relative velocity vector in the inertial frame.

The equations of motion of the KW given in Reference 6 are used for simulation and controller design. The simulation involves a six-degree of freedom KW dynamic model and two two-degree of freedom actuator models for the y and z moving masses. A three degree of freedom particle model is used to simulate the dynamics of the target. North-East-Down (NED) inertial frame of reference is used to represent the position of the KW and the target. Feedback linearization is performed at each instant of time numerically using the nonlinear control system design software. The resulting dynamical system consists of two sixth-order integrators along each control chain with a pseudo-control inputs u_{py} and u_{pz}:

$$\Delta_y^{(6)} = u_{py} \quad \& \quad \Delta_z^{(6)} = u_{pz} \tag{12}$$

The target interception problem can now be posed as a finite-duration linear quadratic control problem on these pure-integrator dynamical systems, with a quadratic performance index consisting of the final values of the relative position components and the integral of a quadratic form in the pseudo-control variables.

The resulting optimal control problem can be solved using the multi-stepping algorithm discussed in Section 2. The solution will be in terms of time-to-go. An approximate value of the time-to-go can be computed using the instantaneous values of the relative velocity and position vectors between the target and the kinetic warhead. Figure 2 shows a block diagram of the closed-loop system. Engagement results using the finite-duration controller are given in the following section.

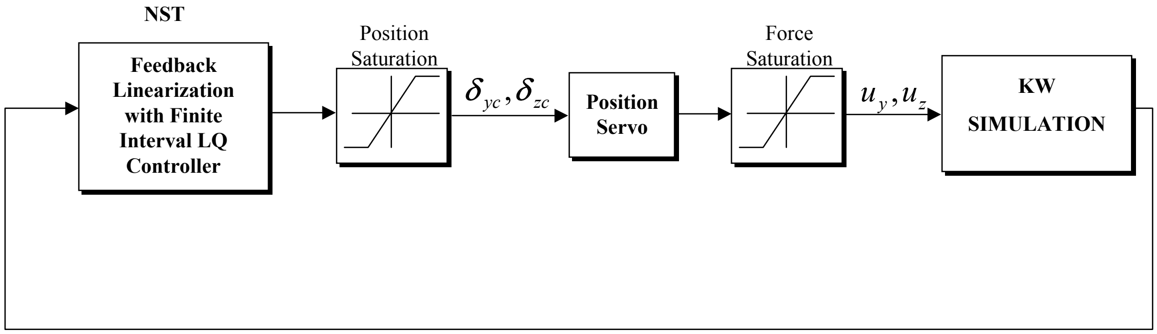

Figure 2. Block Diagram of the Controller Implementation

IV. Engagement Simulations

The following design parameters were chosen for the controller:
*Q = 100*diag([0 0 100 1 0 0])*
R = 1
S_f = *zeros(6, 6);* $S_f(1,1)$ *= 1e6*
n_steps = 20

The parameter *n_steps = 20* refers to the number of intervals employed by the multiple stepping approach to compute the finite interval control.

A. Engagement Scenario #1

Initial conditions on the attitude of the KW are chosen to result in zero angle of attack and zero angle of side-slip at time t = 0. This can be done by choosing the pitch attitude angle same as the flight path angle and the yaw attitude angle same as the heading angle of the KW. The roll attitude angle has always been chosen as zero. The body rates along all three axes have also been chosen as zero at time t = 0. Initial displacements of the moving masses and their initial speeds have also been set to zero.

Shown in Figure 3, Figure 4, and Figure 5 are the trajectories of the KW and the target in 3D, vertical plane and horizontal plane respectively. The target in this case is descending from a higher altitude and is initially located south-east of the KW. The finite duration IGCSD minimizes the y and z components of the relative position vector at the final time as shown in Figure 6. The initial values of the y and z components are zero in the inertial frame owing to the definition of the frame. However, they keep changing due to the difference in the velocity vector and the LOS vector before assuming a very small terminal value. A terminal miss-distance of 0.26481(ft) which is less than the diameter of the KW resulted indicating successful target interception by the KW. The actual and commanded displacements of the y and z actuators are shown in Figure 7 and Figure 8. The commanded displacements are the result of feedback linearization and finite duration control. The y moving mass traverses most of its stroke length whereas the z moving mass travels smaller distance. The magnitude of the mass displacement is governed by the acceleration requirements along the y and z directions. After the first few seconds of transient the attitude of the KW remains constant and the body-rates go to zero as shown in Figure 9. It should be noted that all plots are generated using non-dimensional variables.

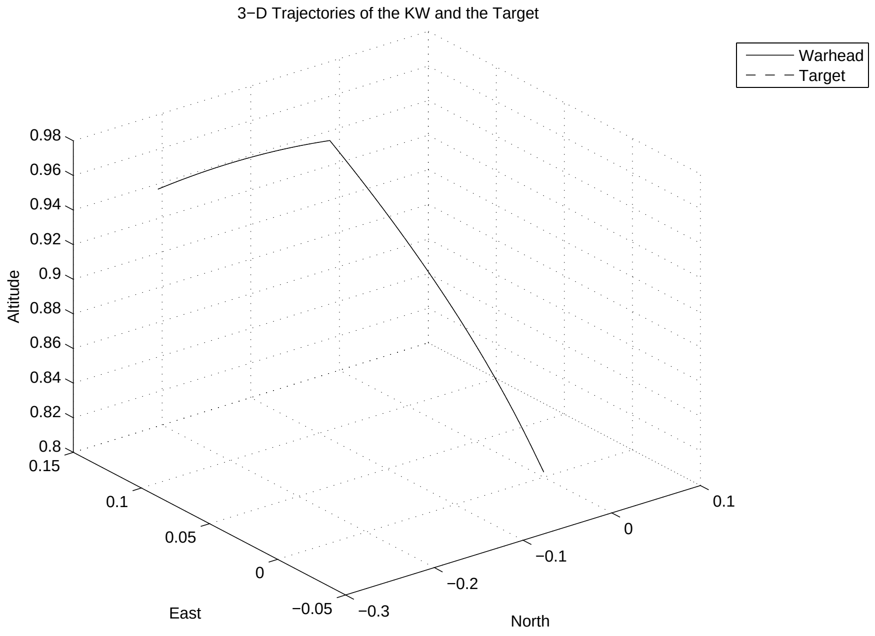

Figure 3. 3D Interception Trajectories

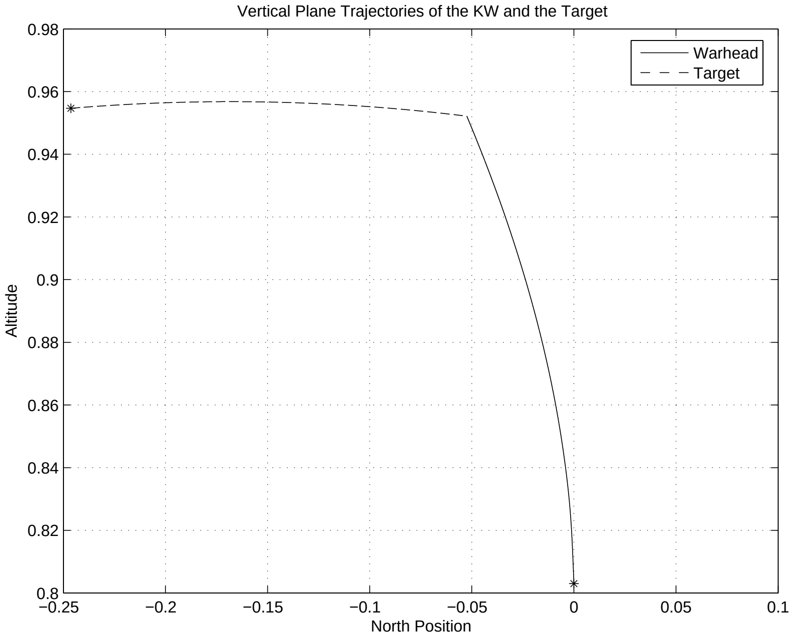

Figure 4. Vertical Plane Trajectories

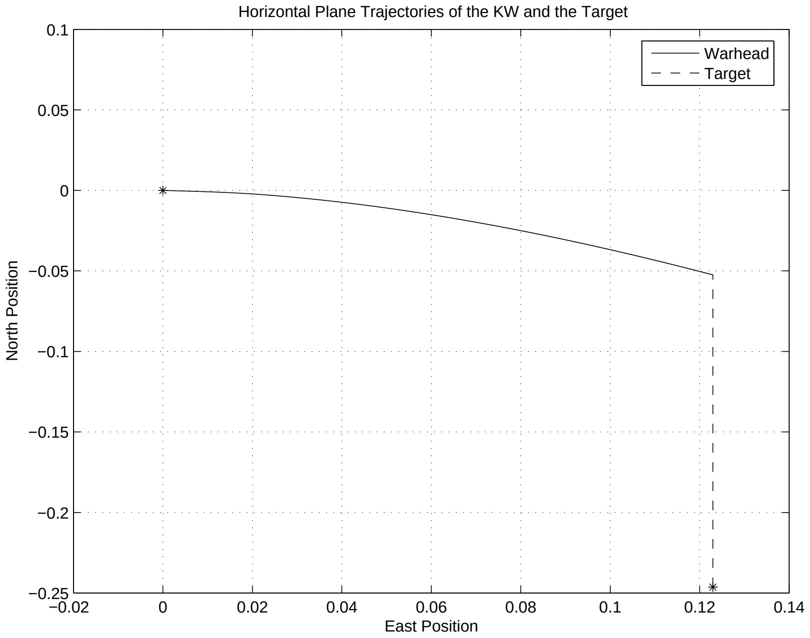

Figure 5. Horizontal Plane Trajectories

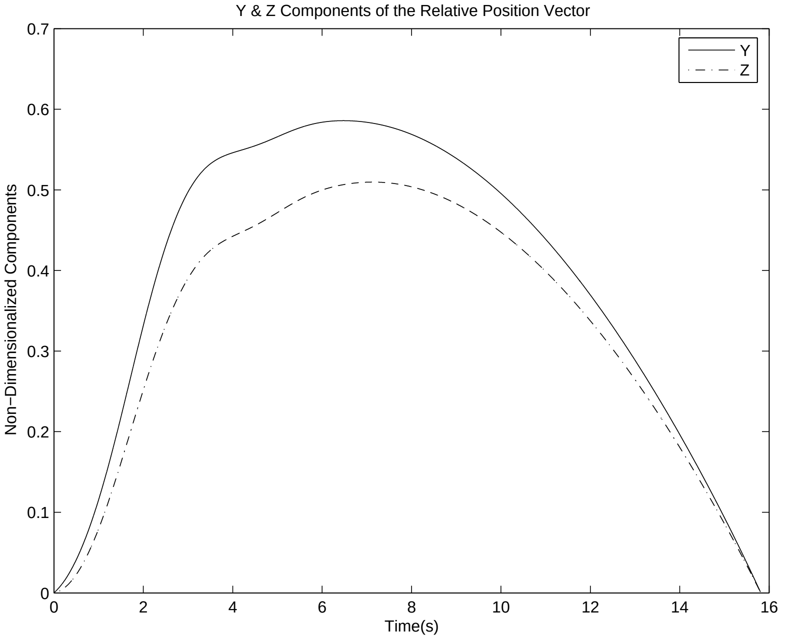

Figure 6. Components of the Relative Position Vector in LOS Frame

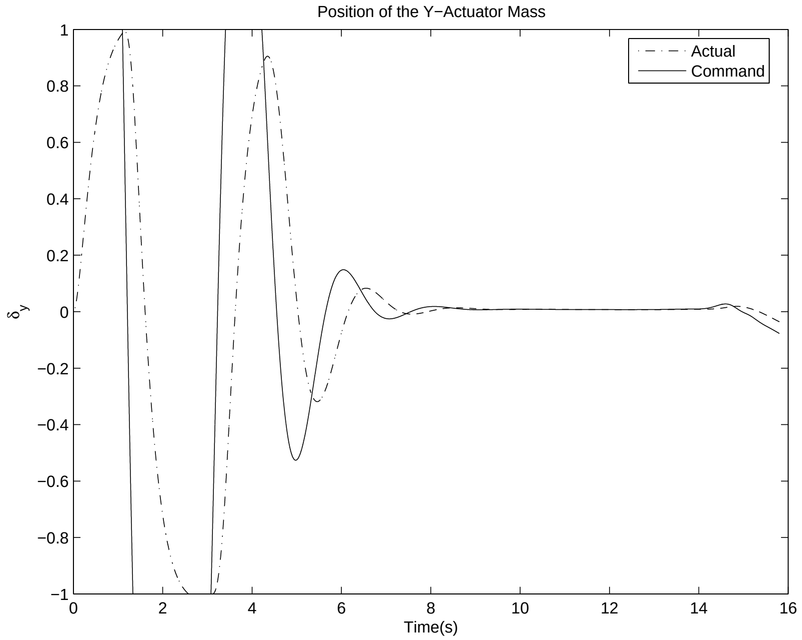

Figure 7. Displacement of the Y-Actuator Moving Mass

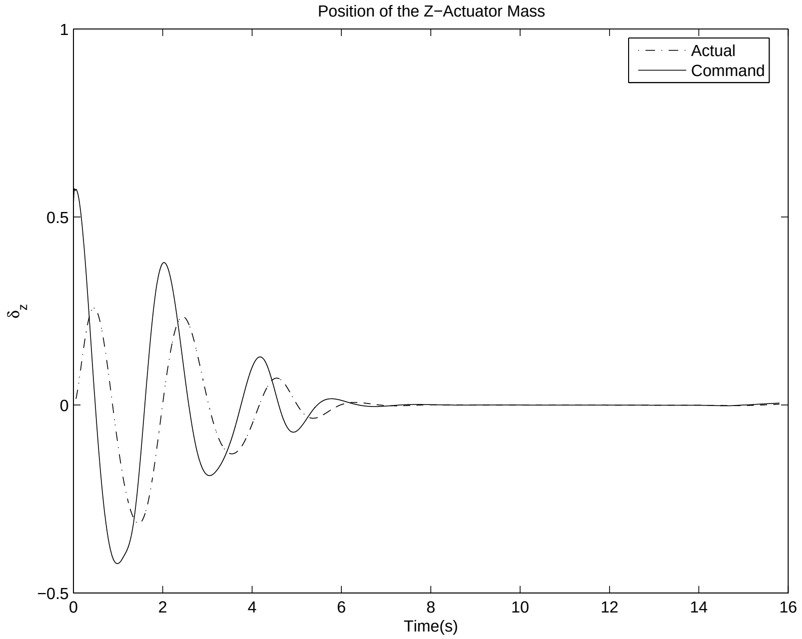

Figure 8. Displacement of the Z-Actuator Moving Mass

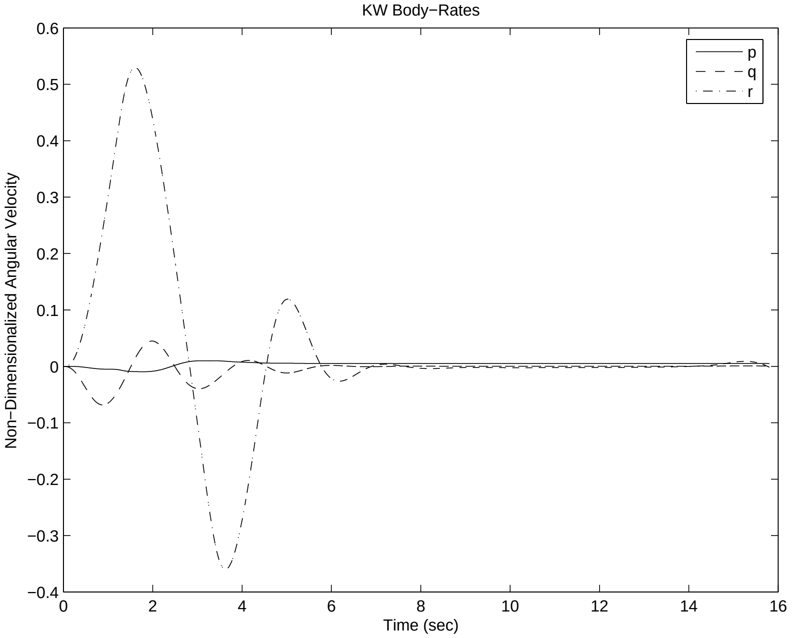

Figure 9. Body Rates of the KW

B. Engagement Scenario #2

The target in this scenario is approaching from the south-west direction of the initial position of the KW. NED frame trajectories of the KW and the target are shown in Figure 10, Figure 11, and Figure 12. Terminal miss-distance obtained from the simulation is 0.22489(ft) indicating successful interception.

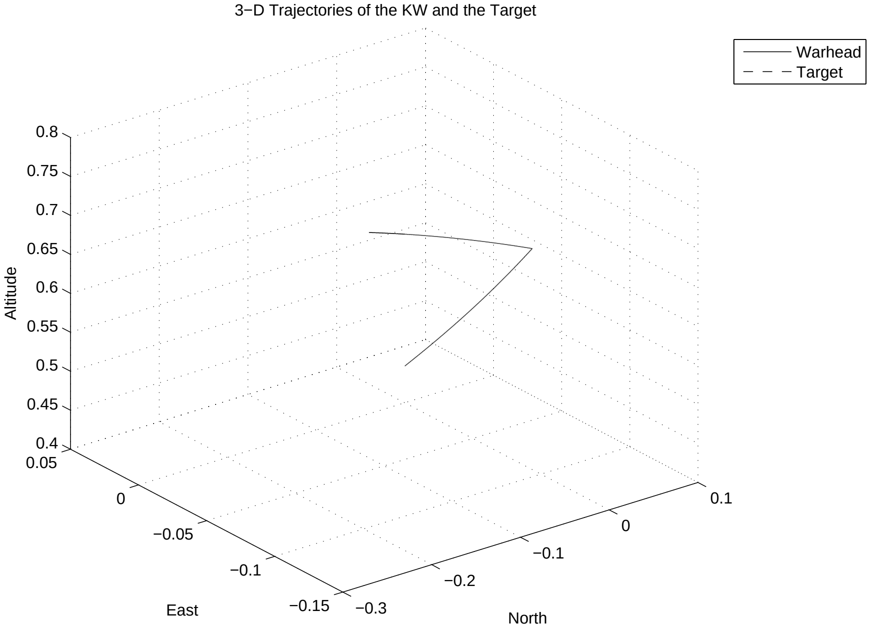

Figure 10. 3D Interception Trajectories

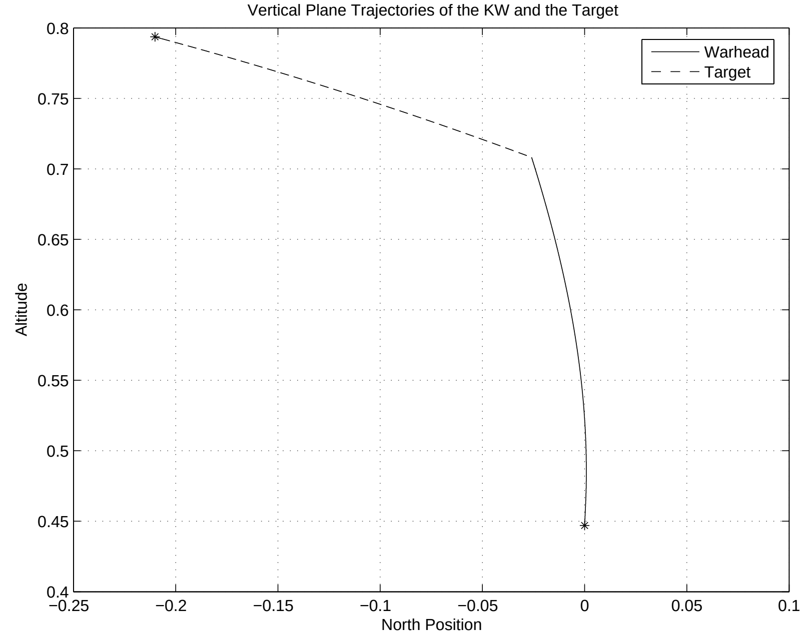

Figure 11. Vertical Plane Trajectories

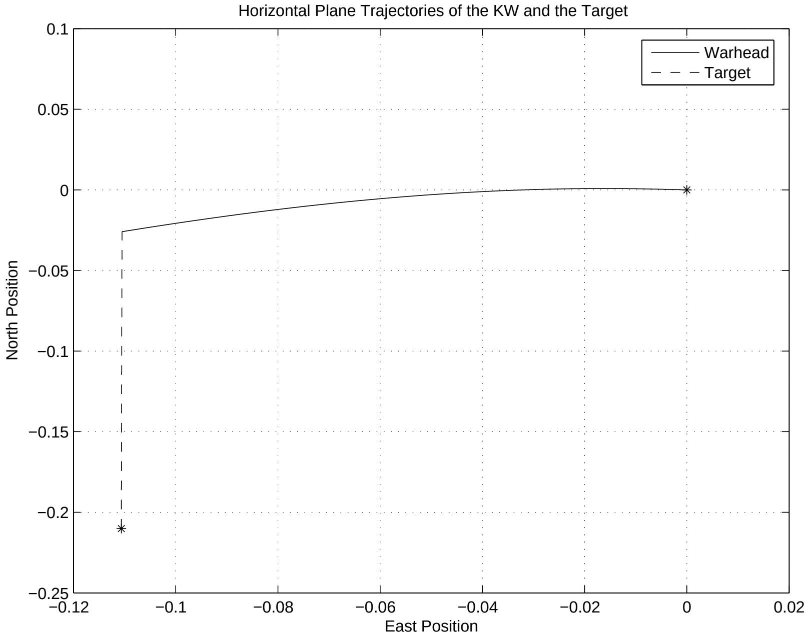

Figure 12. Horizontal Plane Trajectories

Components of the relative vector in the LOS frame are shown in Figure 13 which start and end at zero.

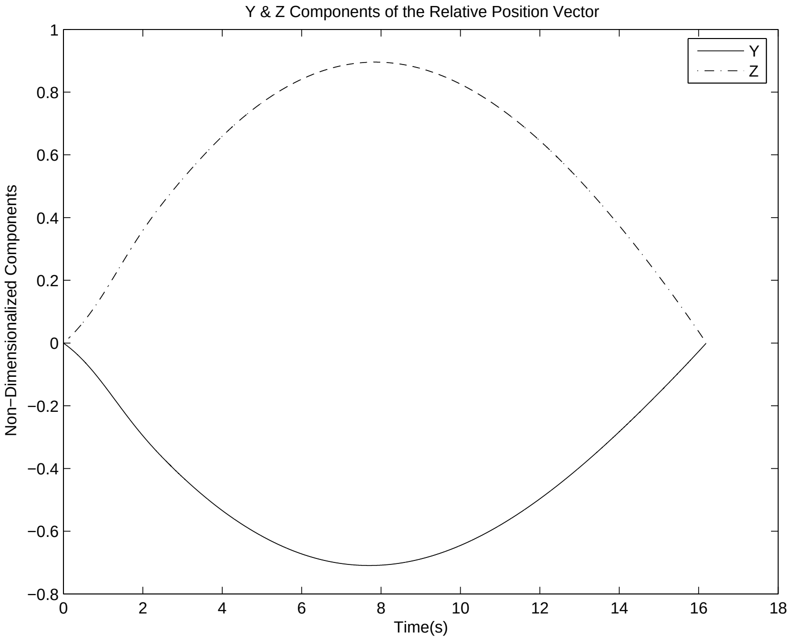

Figure 13. Components of the Relative Position Vector in the LOS Frame

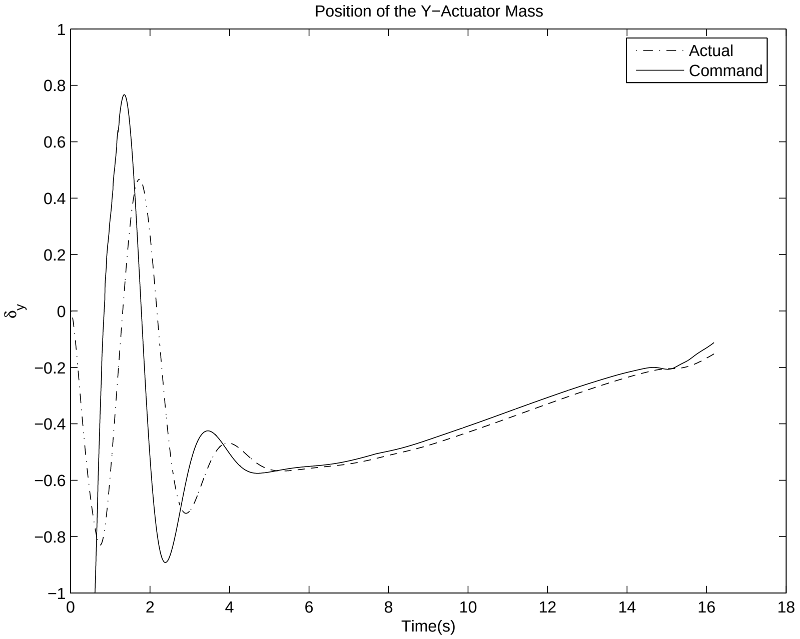

Figure 14. Displacement of the Y-Moving Mass

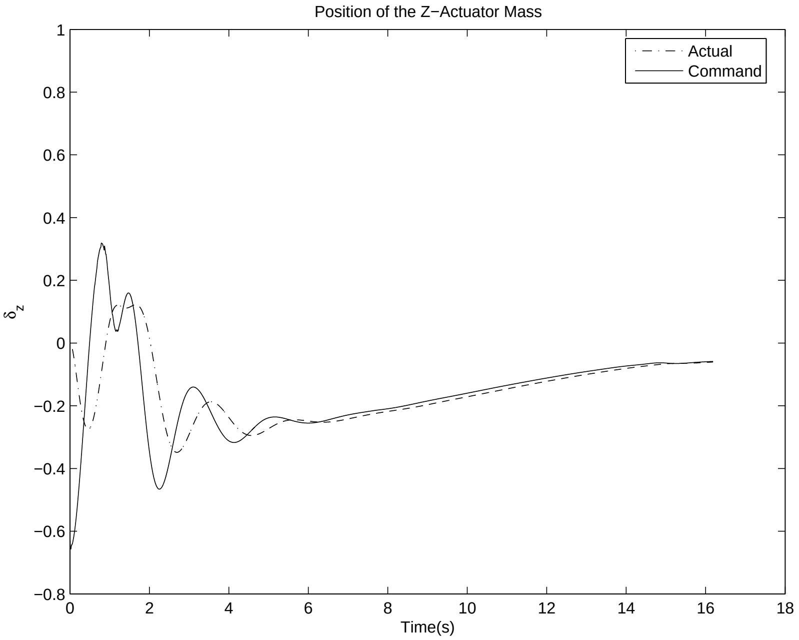

Figure 15. Displacement of the Z-Moving Mass

V. Conclusion

A multi-stepping approach was developed to solve linear two-point boundary value problems arising missile guidance-control problems. This approach is computationally less expensive than the backward integration of the differential Riccatti equation. The algorithm has been applied to a finite-duration integrated guidance-control system design of a moving mass actuated kinetic warhead. Feedback linearization is used to convert the nonlinear control problem into a linear control problem. Finite-interval linear optimal control techniques are then used to pose the control computation problem as a solution to a two-point boundary value problem. Multi-stepping approach presented in this paper has been used to compute the control.

Acknowledgments

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